JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **133**, 573–574 (1988)

A Matrix Trace Inequality

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Submitted by E. Stanley Lee

Received January 13, 1987

In 1978, after he gave some trace inequalities for positive definite matrices, R. Bellman brought attention to two open questions [1]. One of the questions asks: "Is there a matrix analog of the arithmetic mean-geometric mean inequality (for positive definite matrices)?"

In this short note, we prove that the answer to the above question is affirmative.

THEOREM. *If A and B are two $n \times n$ positive definite matrices, then*

(I) $\operatorname{tr}(AB) > 0$ *and*

(II) $\sqrt{\operatorname{tr}(AB)} < (\operatorname{tr} A + \operatorname{tr} B)/2$.

Proof. Let P be an orthogonal matrix such that

$$P'AP = \operatorname{diag}(k_1, ..., k_n) = J.$$

Then

$$\operatorname{tr}(AB) = \operatorname{tr}(P'ABP) = \operatorname{tr}(P'APP'BP) = \operatorname{tr}(JC), \tag{i}$$

where $C = P'BP$ is still a positive definite matrix. Now we have

$$JC = \operatorname{diag}(k_1, ..., k_n) \cdot (c_{ij}) = \begin{pmatrix} k_1 c_{11} & & * \\ & \ddots & \\ * & & k_n c_{nn} \end{pmatrix}.$$

So

$$\operatorname{tr}(JC) = k_1 c_{11} + \cdots + k_n c_{nn} > 0. \tag{ii}$$

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On the other hand, we can easily compute

$$(\operatorname{tr} J + \operatorname{tr} C)^2 - 4 \operatorname{tr}(JC) = (k_1 - c_{11})^2 + \cdots + (k_n - c_{nn})^2 + \text{positive terms} > 0. \tag{iii}$$

(i) and (ii) give (I). From (i) and (iii) we get that

$$\operatorname{tr}(AB) = \operatorname{tr}(JC) < \frac{(\operatorname{tr} J + \operatorname{tr} C)^2}{4} = \left(\frac{\operatorname{tr} A + \operatorname{tr} B}{2}\right)^2. \qquad \text{Q.E.D.}$$

From the proof we see that the same is true for Hermitian matrices.

Reference

1. R. Bellman, Some inequalities for positive definite matrices, in "General Inequalities 2. Proceedings, 2nd Internat. Conf. on General Inequalities," E. F. Beckenbach, Ed.), pp. 89–90, Birkhäuser, Basel, 1980.